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Nonequilibrium quasi-classical effective meson gas: Thermalization

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Abstract. We consider a gas of interacting relativistic effective mesons (qualitatively, like those produced in a heavy-ion collision), regarded as an out-of-equilibrium statistical system. We suppose large occupation numbers, temperature somewhat below typical critical temperatures and the quasi-classical regime. At some initial time t_0, let the gas be in a nonequilibrium state, with spatial inhomogeneities. The time evolution of the gas for $t > t_0$ is studied by a moment method, and appropriate long-time approximations, which could yield the approach to global thermal equilibrium, are discussed.

PACS. 11.10.Wx Finite-temperature field theory – 11.25.Db Properties of perturbation theory – 11.10.Gh Renormalization – 11.90.+t Other topics in general theory of fields and particles

1 Introduction

In heavy-ion collisions, an important issue is how thermal equilibrium is created [1,2]. After a time less than some fm/c, quarks and gluons have thermalized locally, temperatures have fallen a bit below the critical temperatures, and the confinement and chiral phase transitions have occurred. Then, large numbers of pions are produced and, for a time interval between about some fm/c and a few tens of fm/c, before "freeze-out", the pions can be regarded, as least approximately, as a nonequilibrium interacting relativistic quantum gas. We leave aside many important physical features [2] and focus on the time evolution of the pion gas and the eventual formation of a state of more global thermal equilibrium, say, its thermalization. Even leaving aside gauge degrees of freedom and half-integral spin, the analysis is still difficult due to many degrees of freedom out of thermal equilibrium, relativity and quantum aspects. For accounts of nonequilibrium relativistic quantum field theory, see [3,4]. We shall analyze a meson gas, qualitatively similar to, but far simpler than, the pion gas, in the quasi-classical approximation: in so doing, we disregard quantum features, leaving aside their possible or potential relevance, for instance, for Hanbury-Brown and Twiss interferometry [2]. The latter simplification could perhaps be not entirely unreasonable, at least qualitatively, for describing some gross features, in suitably large time and spatial scales, of a nonequilibrium many-meson system, with mesons distributed with large occupation numbers over their quantum states and for a restricted temperature range, analogue to that between the pion mass and the critical temperatures. We accept the possibility that interactions in the meson gas, with an infinite number of degrees of freedom, could give rise to thermalization, and will focus on approximations which could lead to it. We shall deal with the time evolution of the meson gas after some initial time $t = t_0$, analogue of some fm/c, and its eventual approach towards approximate global thermalization for long time, analogue of a few tens of fm/c.

This work is organized as follows. Section 2 deals, for illustrative purposes, with the nonequilibrium statistical mechanics for one degree of freedom in an external "heat bath", and presents the moment technique and the long-time approximation. Section 3 treats one generalization of sect. 2 to an infinite number of degrees of freedom: an effective neutral scalar field. Section 4 outlines some generalization for effective nonlinear chiral fields. Section 5 contains some conclusions and discussions.

2 Oscillator in a "heat bath"

We shall outline the nonequilibrium statistical mechanics of one particle of mass m and momentum p, in one spatial dimension x, with Hamiltonian

$$H = p^2/(2m) + V, \quad V = 2^{-1} m\omega^2 x^2 + (4!)^{-1} g x^4 \quad (2.1)$$

in the presence of a "heat bath" at thermal equilibrium at absolute temperature β_{eq}^{-1}. ω and g are positive constants. The classical probability distribution $W = W(x, p; t)$ for

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the particle fulfills the reversible Liouville equation:

$$\frac{\partial W}{\partial t} = \{H, W\} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial W}{\partial p} \tag{2.2}$$

at time t. $\{H, W\}$ is the classical Poisson bracket. The initial condition at $t = t_0$ is W_{in}. W seems to qualify not just as a classical probability distribution, but also as a quasi-classical one, say, it also accounts for the first correction in Planck's constant, $\hbar$. In fact, let H_Q be the quantum Hamiltonian associated to (2.1), let ρ and $[,]$ be the density operator representing the quantum particle and the commutator. The Schrödinger equation yields:

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H_Q, \rho]. \tag{2.3}$$

Also, let W_Q be the quantum Wigner function determined by ρ [5]. The quantum evolution equation for $\partial W_Q/\partial t$, implied by (2.3), includes in its right-hand-side additive terms of order $\hbar^2$, but no corrections of order $\hbar$ [5]. As $\hbar \to 0$, W_Q and the quantum evolution equation for $\partial W_Q/\partial t$ become W and (2.2), respectively [5]. We shall concentrate on W and (2.2).

Any integration will be performed in $(-\infty, +\infty)$. We shall introduce the following moments W_n $(n = 0, 1, 2, \ldots)$ of W regarding the p-dependence:

$$W_n = W_n(x; t) = \int \mathrm{d}p \frac{H_n((\beta_{eq}/2m)^{1/2}p)}{(\pi^{1/2}2^n n!)^{1/2}} W \tag{2.4}$$

which incorporate the temperature of the "heat bath". H_n is the Hermite polynomial of order n. Equations (2.4) and (2.2) imply the following infinite three-term linear recurrence relation for all W_n's $(n = 0, 1, 2, \ldots, W_{-1} = 0)$:

$$\frac{\partial W_n}{\partial t} = -M_{n,n+1}W_{n+1} - M_{n,n-1}W_{n-1}\,, \tag{2.5}$$

$$M_{n,n+1}W_{n+1} \equiv \left[\frac{(n+1)}{m\beta_{eq}}\right]^{1/2} \frac{\partial W_{n+1}}{\partial x}\,, \tag{2.6}$$

$$M_{n,n-1}W_{n-1} \equiv \left[\frac{n}{m\beta_{eq}}\right]^{1/2} \times \left(\frac{\partial W_{n-1}}{\partial x} + \beta_{eq}\frac{\partial V}{\partial x}W_{n-1}\right). \tag{2.7}$$

The initial condition $W_{n,in}$ is obtained by replacing W by W_{in} in (2.4). A t-independent solution of eq. (2.2) is

$$W_{eq} = \exp[-\beta_{eq}(p^2/(2m) + V)]$$

and, through (2.4), it yields $W_{0,eq}$ proportional to $\exp[-\beta_{eq}V]$ and $W_{n,eq} = 0$, $n = 1, 2, \ldots$. Equation (2.5) implies exactly for any $n \geq 0$

$$\sum_{n'=0}^{n} \int \mathrm{d}x (W_{0,eq})^{-1}[2^{-1}(\partial W_{n'}^2/\partial t) + W_n M_{n,n+1} W_{n+1}] = 0. \tag{2.8}$$

We introduce the Laplace transform:

$$\tilde{W}_n(s) \equiv \int_0^{+\infty} \mathrm{d}t W_n \exp(-st). \tag{2.9}$$

The Laplace transform of (2.5) can be solved formally. That yields all $\tilde{W}_n(s)$, for any $n = 1, \ldots,$ in terms of sums of products of s-dependent linear operators $D[n'; s]$, $n' \geq n$, acting upon $\tilde{W}_{n-1}(s)$ and upon all $W_{n',in}$'s, with $n' \geq n$. The $D[n; s]$'s are infinite continued fractions of products of linear operators, generated by iterating

$$D[n; s] = [s - M_{n,n+1}D[n+1; s]M_{n+1,n}]^{-1}\,. \tag{2.10}$$

For $g = 0$, the harmonic oscillator, the operator $D[n; s]$ (2.10) can be evaluated in closed form, and we shall outline the result. Let

$$y = [m\beta_{eq}/2]^{1/2}\omega x, \qquad A = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}}{\mathrm{d}y} + 2y\right), \tag{2.11}$$

$D[n; s] \equiv D[n; s; A]$ is given by the following fraction:

$$D[n; s; A] = [s + (n+1)\omega^2 A D[n+1; s; A-1]]^{-1}\,. \tag{2.12}$$

Let $f_{n'} = H_{n'}(y)\exp[-2^{-1}y^2]$, $n' = 0, 1, 2 \ldots$. Then, the eigenfunctions of A and of $D[n; s; A]$ are $\exp[-2^{-1}y^2]f_{n'}$. The eigenvalues of A are n' which, through iteration of (2.12), yield directly those of $D[n; s; A]$ as finite fractions, due to the structure $A - 1$. The $D[n; s]$'s cannot be evaluated in closed form for $g \neq 0$.

Thus far, no long-time approximation has been made. We shall analyze the irreversible evolution of the oscillator towards thermal equilibrium, say, its thermalization, induced by the "heat bath". We choose some $n_0(\geq 1)$ and, for $n \geq n_0$, fix $s = \epsilon > 0$ in any $D[n; s]$, ϵ being suitably small. A crucial property, for any $g \geq 0$, is the following: the s-independent $W_{0,eq}^{-1/2}D[n; \epsilon]W_{0,eq}^{1/2}$'s, $n \geq n_0$, are Hermitian operators, have denumerably infinite discrete spectra, without singularities for suitable $\epsilon > 0$, and their eigenvalues have non-negative real parts. Then, the long-time approximation for $n \geq n_0$ is as follows: we replace any $D[n'; s]$ yielding $\tilde{W}_n(s)$, $n \geq n_0$, in terms of $\tilde{W}_{n-1}(s)$ and of $W_{n'',in}$'s, $n'' \geq n$, by $D[n'; \epsilon]$. That approximation is not done for $n < n_0$, and it is the better fulfilled the larger n_0 is. It constitutes a necessary ingredient for the approach towards equilibrium. For a simpler analysis, we also neglect all $W_{n',in}$'s for any $n' \geq n_0$ and set, for small s,

$$\tilde{W}_{n_0}(s) \simeq -D[n_0; \epsilon]M_{n_0,n_0-1}\tilde{W}_{n_0-1}(s). \tag{2.13}$$

The hierarchy becomes closed, by using (2.5), as they stand, for $n = 0, 1, \ldots, n_0 - 1$, and the inverse Laplace transform of (2.13). Its t-independent solution is $W_{0,eq}$ and $W_{n,eq} = 0$, $n = 1, 2, \ldots, n_0$. The solutions of the closed hierarchy relax irreversibly, for $t \gg t_0$ and any reasonable W_{in}, towards the t-independent solution: thermalization of the oscillator due to the "heat bath". For $g \neq 0$ and $\epsilon > 0$, a rough estimate of the matrix elements of $D[n_0; \epsilon]$

over some finite part of its discrete spectrum has been performed: it indicates that, in some average sense, the relaxation times for W_{n_0} are adequately small. Equation (2.8) becomes

$$\sum_{n'=0}^{n_0-1} \int \mathrm{d}x[(2W_{0,eq})^{-1}(\partial W_{n'}^2/\partial t)] \leq 0 \tag{2.14}$$

which also expresses irreversibility due to the "heat bath". In the simplest case, $n_0 = 1$, approximate the linear operator $D[1;\epsilon]$ by a real constant (> 0), and come to (2.5) for $n = 0$. The resulting (irreversible) Fokker-Planck equation for the quasi-classical probability distribution function W_0 is

$$\frac{\partial W_0}{\partial t} = \frac{D[1;\epsilon]}{\beta_{eq}} \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} + \beta_{eq}\frac{\partial V}{\partial x}\right] W_0 \tag{2.15}$$

with the quasi-classical initial condition $W_{0,in}$ at $t = t_0$. For $g = 0$, possible values for the constant $D[1;\epsilon]$ may be estimated qualitatively from (2.12), but a choice for its most reasonable value is open to discussion. The same happens for $g \neq 0$. Equation (2.15) also follows from other different methods [6], and a comparison with them could perhaps help to assess $D[1;\epsilon]$ adequately. This section extends [7], for a discretized spectrum.

3 Quasi-classical effective scalar fields

We shall consider a large statistical system, the dynamics of which is described by a relativistic real scalar classical field χ, having mass parameter m and quartic coupling, in three-dimensional space. m^2 is real. The classical Hamiltonian is, with $\mathbf{x} = (x_1, x_2, x_3)$,

$$H = \int \mathrm{d}^3\mathbf{x}\frac{\pi^2}{2} + V_1 , \tag{3.1}$$

$$V_1 = \int \mathrm{d}^3\mathbf{x}\left\{\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial\chi}{\partial x_i}\right)^2 + \frac{m^2\chi^2}{2} + V(\chi)\right\} , \tag{3.2}$$

$$\pi = \frac{\partial\chi}{\partial t}, \qquad V(\chi) = \frac{g\chi^4}{4!} . \tag{3.3}$$

χ and π are $\mathbf{x}$-dependent, but t-independent, fields. g is the dimensionless coupling constant. An ultraviolet cut-off, Λ, is included in H. χ, m and g are unrenormalized classical quantities. Now, there is no external "heat bath", but the infinite number of degrees of freedom of the classical field will give rise to statistical effects. Let $W = W[\chi, \pi; t]$ be the quasi-classical probability distribution for the system to be described, at time t, by the field configuration χ with momentum π. W fulfills the quasi-classical reversible Liouville equation. The latter reads

$$\frac{\partial W}{\partial t} = \int \mathrm{d}^3\mathbf{x}\left[\left(\frac{\delta V_1}{\delta\chi}\right)\frac{\delta}{\delta\pi} - \pi\frac{\delta}{\delta\chi}\right] W; \tag{3.4}$$

δ/δ denotes the functional derivative. Equation (3.4) generalizes eq. (2.2) to the actual classical-field system. The initial condition is $W_{in} = W_{in}[\chi, \pi]$ at t_0. Let W_{in} be a nonequilibrium state, with spatial inhomogeneities characterized by some function $\beta(\mathbf{x})$: $\beta^{-1}(\mathbf{x})$ could be interpreted, at least qualitatively, as the absolute temperature of the infinitesimal volume $\mathrm{d}^3\mathbf{x}$ at $\mathbf{x}$. With

$$p_0 = p_0(\mathbf{x}) = (2/\beta(\mathbf{x}))^{1/2} ,$$

we shall introduce the functional Hermite polynomials H_n through the Rodrigues-like (functional differentiation) formula:

$$H_n \equiv (-)^n \exp\left[\int \mathrm{d}^3\mathbf{x}\frac{\pi^2}{p_0(\mathbf{x})^2}\right]\frac{\delta}{\delta(\pi(\mathbf{x}_1)/p_0(\mathbf{x}_1))} \times \cdots \frac{\delta}{\delta(\pi(\mathbf{x}_n)/p_0(\mathbf{x}_n))}\exp\left[-\int \mathrm{d}^3\mathbf{x}\frac{\pi^2}{p_0(\mathbf{x})^2}\right] \tag{3.5}$$

thereby generalizing the ordinary Hermite polynomials. H_n depends on $\pi(\mathbf{x}_1)/p_0(\mathbf{x}_1), \ldots, \pi(\mathbf{x}_n)/p_0(\mathbf{x}_n)$. Let $\int[\mathrm{d}\pi]$ denote the functional integration over the classical momentum $\pi(\mathbf{x})$. We shall introduce the moments W_n of W:

$$W_n \equiv \frac{1}{(n!2^n)^{1/2}} \times \int [\mathrm{d}\pi] H_n(\pi(\mathbf{x}_1)/p_0(\mathbf{x}_1), \ldots, \pi(\mathbf{x}_n)/p_0(\mathbf{x}_n))W. \tag{3.6}$$

The W_n's depend on $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and, although not written explicitly, also on χ and t. Like (2.2) via (2.4), (3.4) gives, through (3.6), the following (reversible) infinite linear hierarchy for all W_n's ($n = 0, 1, \ldots, W_{-1} \equiv 0$):

$$\frac{\partial W_n}{\partial t} = -M_{n,n+1}W_{n+1} - M_{n,n-1}W_{n-1} , \tag{3.7}$$

$$M_{n,n+1}W_{n+1} \equiv \left[\frac{n+1}{2}\right]^{1/2}\int \mathrm{d}^3\mathbf{x}p_0(\mathbf{x})\frac{\delta}{\delta\chi(\mathbf{x})} \times W_{n+1}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n), \tag{3.8}$$

$$M_{n,n-1}W_{n-1} \equiv \int \mathrm{d}^3\mathbf{x}\frac{p_0(\mathbf{x})}{(2n)^{1/2}}FP(\mathbf{x})\left[\sum_{i=1}^{n}\delta^{(3)}(\mathbf{x}-\mathbf{x}_i) \times W_{n-1}(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)\right], \tag{3.9}$$

$$FP(\mathbf{x}) = \frac{\delta}{\delta\chi(\mathbf{x})} + \frac{2}{p_0(\mathbf{x})^2}\frac{\delta V_1}{\delta\chi(\mathbf{x})} . \tag{3.10}$$

The initial condition $W_{n,in}$ for (3.7) is obtained by replacing W by W_{in} in (3.6). In particular, the equations in (3.7) for $n = 0, 1, 2$ can be shown to be exactly consistent with the balance equations for momentum, energy and angular momentum.

Like in sect. 2, we introduce the Laplace transform $\tilde{W}_n(s)$ of W_n, solve formally the Laplace transform of (3.7) and get all $\tilde{W}_n(s)$, for any $n = 1, \ldots,$ in terms of similar structures and new s-dependent linear operators $D[n'; s]$. The $D[n; s]$'s are infinite continued fractions formally similar to (2.10).

3.1 Initial state not far from equilibrium

Let W_{in}, representing also nonequilibrium, be not far from global thermal equilibrium at constant absolute temperature β_{eq}^{-1}. For instance: i) $\beta(\mathbf{x}) = \beta_{eq} + \delta\beta(\mathbf{x})$, where the function $\delta\beta(\mathbf{x}) \to 0$ for $|\mathbf{x}| \to +\infty$ (and $|\delta\beta(\mathbf{x})| \leq \beta_{eq}$), or ii) the larger parts of the gas are at global thermal equilibrium at β_{eq}^{-1}, while the remaining, smaller, parts are, still, out of equilibrium, with spatial inhomogeneities. β_{eq}^{-1} is not imposed by any external "heat bath", but by the infinite number of degrees of freedom, not far from thermal equilibrium, of the whole field system. Then, we replace $p_0(\mathbf{x})$ by the constant $p_0 = (2/\beta_{eq})^{1/2}$ in (3.5)–(3.10), and until otherwise stated. Then, $W_{0,eq} = \exp[-\beta_{eq} \int \mathrm{d}^3\mathbf{x} V_1(\chi)]$ and $W_{n,eq} = 0$, $n \geq 1$ yield a t-independent solution of (3.7). Equation (3.7) implies exactly the analogue of (2.8). With the actual W_{in}, the long-time approximation, with $D[n_0; s] \simeq D[n_0; \epsilon]$ for $n \geq n_0$, proceeds formally like in sect. 2, and it is the better fulfilled the larger n_0 is. The approximation seems justified due to the integrations and functional dependences involved in (3.8) and (3.9), say, to the infinite number of degrees of freedom of the large statistical system. Then, $W_{0,eq}^{-1/2} D[n;\epsilon] W_{0,eq}^{1/2}$'s, $n \geq n_0$, now have continuous spectra. In principle, the operator $D[n';\epsilon]$ depends on g and on the dimensionful quantities ϵ, m, Λ and β_{eq}^{-1}. The analogue of (2.14) also holds. For $n_0 = 1$, by proceeding like in sect. 2, the resulting (irreversible) Fokker-Planck equation for the quasi-classical probability distribution functional W_0 is

$$\frac{\partial W_0}{\partial t} = \frac{D[1;\epsilon]}{\beta_{eq}} \int \mathrm{d}^3\mathbf{x} \frac{\delta}{\delta\chi(\mathbf{x})} FP(\mathbf{x}) W_0 \qquad (3.11)$$

with the quasi-classical initial condition $W_{0,in}$ at $t = t_0$. $\chi = \chi(\mathbf{x})$ plays the role of an order parameter. The t-independent solution of (3.11) is $W_{0,eq}$. The ansatz has been made of interpreting $D[1;\epsilon]$ as a positive constant, instead of as an operator. Then, the solutions of (3.11) relax irreversibly, for $t \gg t_0$ and any reasonable $W_{0,in}$, towards $W_{0,eq}$: thermalization. Physically, the larger parts of the gas, already at, or close to, global thermal equilibrium at β_{eq}^{-1} at t_0, iron out all spatial inhomogeneities and drive the remaining, smaller, parts also to the same global equilibrium distribution at β_{eq}^{-1}, $W_{0,eq}$, for long times.

3.2 Removing the cut-off

With the same W_{in} as in subsect. 3.1, let us now remove the ultraviolet cut-off in (3.11): $\Lambda \to +\infty$. We apply results in [8,9] to (3.11). With the quartic self-coupling in eq. (3.3), $W_{0,eq}$ characterizes a superenormalizable theory in three spatial dimensions. In $W_{0,eq}$ and in the dynamical theory described by (3.11) for $t > t_0$, mass renormalization is necessary but neither χ nor g require ultraviolet renormalization. The question then arises whether the constant $D[1;\epsilon]$ would require it: if that were the case, that would signal some physical failure in the long-time approximation. It turns out that $D[1;\epsilon]$ does not require any ultraviolet renormalization either. Then, the dynamical theory described by (3.11) for $t > t_0$ is also superenormalizable. Physically, $D[1;\epsilon]$, being related to long-time and large-distance behaviours, should remain finite as $\Lambda \to +\infty$. This indicates, *a posteriori*, that the long-time approximation does not run into conflict with the ultraviolet behaviour, at least in the actual three-dimensional classical model. The above analysis for $n_0 = 1$ suggests the following, for $n_0 > 1$. The approximate dynamical theory given by the analogue of (2.13) with $D[n_0;\epsilon]$ interpreted as a positive constant, and by (3.7) for $n < n_0$ would be superenormalizable and would relax for a long time towards the same $W_{0,eq}$ as above. $D[n_0;\epsilon]$ would require no ultraviolet renormalization, only m being in need of it.

3.3 Initial state far from equilibrium

Now, let the gas be at t_0 in a state W_{in} quite appreciably out of global thermal equilibrium, say, with spatial inhomogeneities so that $\beta(\mathbf{x})$ is neatly different from any β_{eq}. Now, we deal with (3.7) with $p_0 = p_0(\mathbf{x})$. The long-time approximation also proceeds (formally, at least) by setting $s = \epsilon$ in any $D[n';s]$ for $n' \geq n \geq n_0 (\geq 1)$. The hermiticity and positivity properties of $D[n';\epsilon]$ in subsect. 3.1 no longer hold necessarily, due to the $\mathbf{x}$-dependence of p_0. Equations (2.8) and (2.14) do not hold necessarily. Anyway, simplifying assumptions similar to those yielding (3.11) now give (at least, formally and for fixed Λ) the generalized Fokker-Planck equation:

$$\frac{\partial W_0}{\partial t} = D[1;\epsilon] \int \mathrm{d}^3\mathbf{x} \frac{p_0(\mathbf{x})}{2} \frac{\delta}{\delta\chi(\mathbf{x})} \times p_0(\mathbf{x}) FP(\mathbf{x}) W_0 \qquad (3.12)$$

with the quasi-classical initial condition $W_{0,in}$ and a real and positive constant $D[1;\epsilon]$, so that irreversibility holds. As $p_0 = p_0(\mathbf{x})$, the actual equilibrium distribution $W_{0,eq}$ is not given by $FP(\mathbf{x})W_{0,eq} = 0$, but as the limit of the solution W_0 of (3.12) for $t \to +\infty$: such $W_{0,eq}$ would require longer times to be reached and depend on some global equilibrium temperature ($\neq p_0(\mathbf{x})^2/2$). See [8], for instance, for functional integral representations of $W_{0,eq}$.

4 Quasi-classical effective chiral fields

Let the meson gas be a nonequilibrium (quasi-classical) large statistical system, described by the simplest ($O(N)$-invariant) nonlinear σ model [8]. Let $\chi_i = \chi_i(\mathbf{x})$, $i = 1, \ldots, N-1$, be classical effective nonlinear chiral fields for mesons and $\pi_i = \pi_i(\mathbf{x})$ be their associated momenta. The classical Hamiltonian is

$$H = \int \mathrm{d}^3\mathbf{x} h_\sigma + V_1 \,, \qquad (4.1)$$

$$h_\sigma = \frac{g_\sigma^2}{2} \sum_{i,j=1}^{N-1} \pi_i (G^{-1})_{i,j} \pi_j \,, \qquad (4.2)$$

$$V_1 = \frac{g_\sigma^2}{2} \int \mathrm{d}^3\mathbf{x} \sum_{i,j=1}^{N-1} \left[G_{ij} \sum_{l=1}^{3} \left(\frac{\partial\chi_i}{\partial x_l} \frac{\partial\chi_j}{\partial x_l} \right) \right] , \qquad (4.3)$$

$$G_{ij} = \delta_{i,j} + \frac{\chi_i \chi_j}{1 - \bar{\chi}^2} \,. \qquad (4.4)$$

The coupling constant g_σ is now dimensionful. G^{-1} is the inverse of the $(N-1)\times(N-1)$ matrix formed by all G_{ij}. An ultraviolet cut-off about or somewhat larger than g_σ is supposed. The role played in sect. 3 by the functional Hermite polynomials H_n will now be played by new functional polynomials $H_{\sigma,n}$ which, by definition, are orthogonalized with respect to the functional measure

$$\int \prod_{i=1}^{N-1} [\mathrm{d}\pi_i] \exp\left[-\int \mathrm{d}^3\mathbf{x} 2p_0(\mathbf{x})^{-2} h_\sigma\right].$$

$H_{\sigma,n}$ would allow to introduce moments and to generalize formally the developments and results in sect. 3. Thus, if the initial nonequilibrium distribution at t_0 is not far from global thermal equilibrium at constant absolute temperature β_{eq}^{-1}, the counterpart of (3.11) reads for long times

$$\frac{\partial W_0}{\partial t} = \frac{D[1;\epsilon]}{\beta_{eq}} \sum_{l=1}^{N-1} \int \mathrm{d}^3\mathbf{x} \frac{\delta}{\delta\chi_i(\mathbf{x})} \sum_{j=1}^{N-1} (G^{-1})_{i,j} \times FP_j(\mathbf{x}) W_0\,, \tag{4.5}$$

$$FP_j(\mathbf{x}) = \frac{\delta}{\delta\chi(\mathbf{x})_j} + \beta_{eq}\frac{\delta V_1}{\delta\chi(\mathbf{x})_j} + \frac{\delta \ln[\det G^{-1}]^{1/2}}{\delta\chi(\mathbf{x})_j}. \tag{4.6}$$

$[\det G^{-1}]$ denotes the determinant of G^{-1} (arising from $\int\prod_{i=1}^{N-1}[\mathrm{d}\pi_i]$), both as a $(N-1)\times(N-1)$ matrix determinant and as a functional determinant. $D[1;\epsilon]$ is a positive constant. Equation (4.6) has the t-independent solution

$$W_{0,eq} = [\det G^{-1}]^{-1/2} \exp\left[-\beta_{eq}\int \mathrm{d}^3\mathbf{x} V_1\right],$$

to which W_0 relaxes (at least, formally) for long times. For fixed ultraviolet cut-off, the analysis in subsect. 3.1 could apply in principle, but $[\det G^{-1}]$, $W_{0,eq}$ and (4.6) would require a detailed regularization and study, through techniques related to those employed for the quantum nonlinear σ model [8]. See [10] for the dynamics of a relaxational nonlinear σ model, for cooperative phenomena.

5 Conclusion and discussion

We have treated a nonequilibrium meson gas, through quasi-classical effective fields, as a caricature of the pion gas produced in a heavy-ion collision, say, between the phase transitions and "freeze-out". For simplicity, we started out with one degree of freedom and, later, turned to an infinite number of degrees of freedom: a scalar field and nonlinear chiral fields. At some initial time t_0, the meson gas is out of global thermal equilibrium. Neither transport theory nor Kubo formulae have been invoked. The temporal evolution of the gas has been analyzed through three-term linear hierarchies for moments of probability distributions. They provide an adequate framework to infer that, due to the infinite number of degrees of freedom involved and under suitable long-time approximations: a) higher-order moments follow adiabatically the dynamics of the lower ones, and b) lower-order moments drive the relaxation towards approximate global thermal equilibrium. Not far from global thermal equilibrium, certain Hermiticity and positivity properties of the operators $D[n;\epsilon]$ play a crucial role. Those techniques seem, so far, less efficient for direct quantitative estimates of relaxation times: for this reason, one would require explicit approximations for the $D[n;\epsilon]$'s, outside our scope here.

In [3], the nonequilibrium quantum generating functional Z associated to (3.1)–(3.3) has been analyzed for long times, with several quasi-classical assumptions and approximations. Then, Z becomes approximately the nonequilibrium generating functional for a purely dissipative quasi-classical functional Fokker-Planck process, for a suitable order parameter. The nonperturbative quasi-classical and long-time approximations in sect. 3 are quite different from those in [3], but both [3] and the present work lead, essentially and consistently, to equivalent Fokker-Planck dynamics. The time evolution and long-time thermalization in classical-field theories has been investigated by other different methods: see [4,11–15].

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